Relaxation in Perturbed Area-Preserving Maps

W. Breymann
Institut für Physik, Universität Basel, Basel, Switzerland

Z. Naturforsch. **48a**, 663–665 (1993); received Februar 16, 1993

By means of the linear response of the cat map to a constant perturbation, it is illustrated how relaxation to equilibrium occurs in chaotic hamiltonian systems.

When a chaotic map is perturbed ...

How does relaxation to equilibrium occur in chaotic hamiltonian systems? The answer to this question is illustrated by means of the linear response to an external perturbation of a two-dimensional area-preserving map. Such maps are the simplest models for hamiltonian systems. As in [1], the unperturbed system will be Arnold's cat map [2], namely

$$f\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \bmod 1\,. \qquad (1)$$

It is completely chaotic and has the positive Lyapunov exponent $\lambda_+ = (3+\sqrt{5})/2 \approx 0.962$. For sufficiently small perturbations, the perturbed map can always be written as a conjugation with an appropriate perturbative function g_α [3]: $\hat{f}_\alpha = \hat{g}_\alpha \circ f \circ \hat{g}_\alpha^{-1}$, where the value of α indicates the strength of the perturbation, and $g_0 = \mathrm{id}$ is the identity function. We choose a specific perturbation which acts on the x-coordinate only: $[g_\alpha]_y(x, y) = y$ and

$$[g_\alpha]_x(x, y) = \begin{cases} (1+2\alpha)\,x & \text{if} \quad x \le 1/2\,, \\ (1-2\alpha)\,x + 2\alpha & \text{if} \quad x > 1/2\,. \end{cases} \qquad (2)$$

For sufficiently small α, $\hat{f}_\alpha$ has the same chaotic properties as f. An orbit of the perturbed map with initial condition x_0 is given by $\hat{x}_t = \hat{f}_\alpha^t(x_0)$. Due to the positive Lyapunov exponent, it diverges from the unperturbed one exponentially in time.

... the response of the state space density ...

For the unperturbed map (1), the time evolution of the state-space density is determined by the Frobenius-Perron (FP) operator $\mathscr{L}$ which transforms the density ϱ_t at time step t into the density ϱ_{t+1} at time step $t+1$ [4]. If the system was in equilibrium for $t < 0$ and the perturbation has been switched on at $t=0$, then the density $\hat{\varrho}_t$ of the perturbed system is obtained by t-fold application of the FP-operator $\hat{\mathscr{L}}_\alpha$ associated with the perturbed map $\hat{f}_\alpha$ to the unperturbed invariant density: $\hat{\varrho}_t = \hat{\mathscr{L}}_\alpha^t \varrho^*$. To calculate the response $\delta\hat{\varrho}_t = \delta\hat{\mathscr{L}}_\alpha^t \varrho^*$ of the state space density in linear order, the perturbation $\delta\hat{\mathscr{L}}_\alpha^t = (\hat{\mathscr{L}}_\alpha^t - \mathscr{L}^t)$ of the time-evolution operator is approximated to first order in α [1]:

$$\delta\hat{\mathscr{L}}_\alpha^t \simeq \alpha \sum_{s=1}^{t} \mathscr{L}^{t-s}\,\hat{\mathscr{L}}^{(1)}\,\mathscr{L}^{s-1}. \qquad (3)$$

$\mathscr{L}\varrho^* = \varrho^*$, and $\hat{\mathscr{L}}^{(1)}\varrho^*$ can be evaluated explicitly in terms of the perturbative function $g_{\alpha=1}$:

$$\begin{aligned}(\hat{\mathscr{L}}^{(1)}\varrho^*)\,(x, y) &= ((\mathrm{Id} - \mathscr{L})\,\mathrm{div}(g_1 - \mathrm{id}))(x, y) \\ &= (\mathrm{Id} - \mathscr{L})\,\mathrm{sign}(x - 1/2)\,, \end{aligned} \qquad (4)$$

where the second equality holds for the perturbation (2). The action of the FP operator $\mathscr{L}^t$ on a density ϱ reads

$$(\mathscr{L}\varrho)(x, y) = \varrho(f^{-t}(x, y))\,. \qquad (5)$$

For the – unperturbed – cat map, (5) can be evaluated explicitly in Fourier space:

$$\mathscr{L}^t\, e^{2\pi i (kx+ly)} \qquad (6)$$
$$= \exp\{2\pi i\,[(k\mathscr{F}_{2t+1} - l\mathscr{F}_{2t})\,x + (-k\mathscr{F}_{2t} + l\mathscr{F}_{2t-1})\,y]\}\,.$$

Here, k, l are integers and $\mathscr{F}_n$ labels the n-th Fibonacci number. The fact that a basis function of the Fourier space remains a basis function under the action of $\mathscr{L}^t$ considerably facilitates the further analysis.

Taking into account (3), (4) and (6), the time evolution of the density response to the perturbation (2) reads to first order in α

$$\delta\hat{\varrho}_t(x, y) = (\delta\hat{\mathscr{L}}_\alpha^t \varrho^*)(x, y) \simeq \frac{i\alpha}{\pi} \sum_k \frac{1-(-1)^k}{k} \qquad (7)$$
$$\cdot\,[\exp\{2\pi i k(\mathscr{F}_{2t+1}\,x - \mathscr{F}_{2t}\,y)\} - \exp\{2\pi i k x\}]\,.$$

Reprint requests to Dr. W. Breymann, Institut für Physik, Universität Basel, Klingelbergstraße 82, CH-4056 Basel, Schweiz.

0932-0784 / 93 / 0500-0663 $ 01.30/0. – Please order a reprint rather than making your own copy.

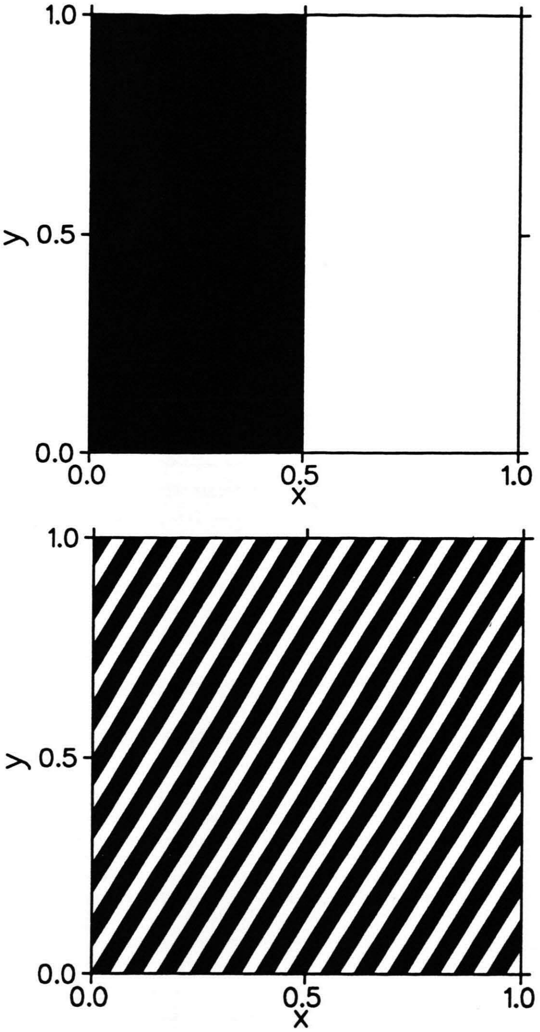

Fig. 1. Linear response $\delta\hat{\varrho}_t$ of the state-space density due to the perturbation $g_{\alpha=1}$ defined in (2), at time (a) $t=0$ and (b) $t=3$. Black: $\delta\hat{\varrho}_t(x, y) = -1$, white: $\delta\hat{\varrho}_t(x, y) = +1$.

... does not relax to equilibrium ...

The FP operator $\mathscr{L}$ is unitary. Its spectrum consists of the single non-degenerate eigenvalue 1 with eigenfunction $\varrho^*(x, y) = 1$, and an infinite set of eigenvalues, all of modulus 1, with δ-distributions as eigenfunctions. ϱ^* is the unique invariant density. However, an arbitrary density response $\delta\hat{\varrho}$ cannot relax to zero in the ordinary sense: Let $\| \;\|$ be any p-norm, then it follows easily from (5) that $\|\delta\hat{\varrho}_t\|$ remains invariant under the action of the FP-operator, $\|\delta\hat{\varrho}_t\| = \|\mathscr{L}^t \delta\hat{\varrho}\| = \|\delta\hat{\varrho}\|$, for all t (cf. Figure 1).

... but the mean values of observables do ...

Even though density perturbations do not relax to equilibrium this does not imply that perturbations of the mean values of observables will relax neither. Observables B are state-space functions $B(x)$, and their mean response $\langle\delta B\rangle_{\hat{\varrho}}$ to a perturbation is given by a weighted average over the whole space, the weighting function being the density response $\delta\hat{\varrho}_t$:

$$\langle\delta B\rangle_{\hat{\varrho}}(t) = \int_M B(x)\, \delta\hat{\varrho}_t(x)\, \mathrm{d}x\,. \tag{8}$$

If $\hat{\varrho}_t$ is an L^1-function, B has to be an L^∞-function [4]. Using (7) yields the linear response of the observable in terms of the perturbative function g_1:

$$\langle\delta B\rangle_{\hat{\varrho}}(t) = -\frac{i\alpha}{\pi}\sum_m \frac{1-(-1)^m}{m} \cdot \left(\tilde{B}(m, 0) - \tilde{B}(m\mathscr{F}_{2t+1}, -m\mathscr{F}_{2t})\right). \tag{9}$$

In this expression, $\tilde{B}(k, l)$ denotes the (k, l)-th Fourier component of the observable B.

The response splits into a stationary part (the first term in the big parenthesis of (9)) and a time-dependent part (second term in the parenthesis). The latter, $(i\alpha/\pi)\sum_m (1-(-1)^m)/m\, \tilde{B}(m\mathscr{F}_{2t+1}, -m\mathscr{F}_{2t})$, relaxes to zero because for increasing t, higher and higher Fourier components are essential for the response, and $\tilde{B}(k, l)$ tends to zero sufficiently fast when $|k|$ or $|l|$ tends to infinity. The first property is the manifestation of mixing in Fourier space, and the second one follows from the fact that $B \in L^\infty$. Consider, e.g., the observable $B = x y^2$, the linear response of which is given by

$$\langle\delta(x y^2)\rangle_{\hat{\varrho}}(t) = \frac{\alpha}{12}\left(1 - \frac{1}{\mathscr{F}_{2t+1}\mathscr{F}_{2t}^2}\right) \simeq \frac{\alpha}{12}\left(1 - C \cdot e^{-3\lambda_+ t}\right). \tag{10}$$

The last line holds asymptotically for large t, and the constant C takes the value $5\sqrt{5}/(8(1+\sqrt{5}))$. This observable clearly relaxes and furthermore, the relaxation time is proportional to the inverse of the positive Lyapunov exponent of the unperturbed map. Thus, *mean values* of observables *do* decay to their equilibrium values when time tends to infinity, and in a K system this happens in an exponential fashion even though the *norm* of $\delta\varrho$ does *not* converge to zero. The mathematical point is that convergence in the norm is not necessary in order to assure relaxation of mean values, *weak* convergence is sufficient.

... due to coarse graining and randomization

The mathematical statement about convergence can be expressed in the language of physics. But first, let us look in some more detail at what happens to a perturbation during the time evolution described by (9). Plane waves $\exp\{2\pi i(kx+ly)\}$ with (k, l) such that $k^2+kl-l^2=$ const are related through (6). Unter time evolution, k and $-l$ tend to infinity simultaneously. In other words, the perturbed density wrinkles more and more in the course of time (cf. Figure 1). This behaviour of $\delta\hat{\varrho}_t$ allows to estimate the relaxation time of the linear response of an observable B. Assuming that the derivative DB is an L^∞-function, the average $\langle\delta B\rangle(t)$ should relax at least as $2\,\|DB\|_\infty\cdot l(t)$, where $l(t)$ is the width of a black stripe in Figure 1. Explicit calculation shows that asymptotically $l(t)\propto e^{-\lambda_+ t}$: The time constant for relaxation should be smaller or equal to the inverse of the positive Lyapunov exponent of the unperturbed map.

Randomization and *coarse graining* can be recognized as the mechanism giving rise to the relaxation property. Due to the mixing dynamics, the time evolution leads to *randomization:* the term $\mathscr{L}^t\,\mathrm{sign}(x-1/2)$ will vary over shorter and shorter length scales when t increases. *Coarse graining* is introduced implicitly by the length scale over which $B(x)$ varies significantly. The relaxation time is then expected to be the time necessary to transform a fluctuation on the length scale of the domain of the unperturbed map into one on the length scale of the implicit coarse graining associated with the observable B. Mixing dyamics is the necessary and sufficient condition for relaxation behaviour.

The author would like to thank H. Thomas for many fruitful discussions. The financial support of the Swiss National Science Foundation is gratefully acknowledged. The ideas in this paper have been exposed at the Third Annual Meeting of ENGADYN, "Workshop on Nonlinearities, Dynamics, and Fractals", Oct. 12–Oct. 15, 1992 in Grenoble.

[1] W. Breymann and H. Thomas, in: From Phase Transitions to Chaos (G. Györgyi et al., eds.), World Scientific, Singapore 1992, p. 335.
[2] V. I. Arnold, and A. Avez, Ergodic Problems of Classical Mechanics, W. A. Benjamin, New York 1968.
[3] V. I. Arnold, Geometrische Methoden in der Theorie der gewöhnlichen Differentialgleichungen, Sect. 3.4, Birkhäuser, Basel 1987.
[4] A. Lasota and M. Mackey, Probabilistic properties of deterministic systems, Cambridge University Press, Cambridge 1985.